Acta Mech 205, 161–170 (2009)
DOI 10.1007/s00707-009-0169-0

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Extremum and variational principles for elastic and inelastic media with fractal geometries

Received: 25 June 2008 / Revised: 16 September 2008 / Published online: 7 April 2009

Abstract This paper further continues the recently begun extension of continuum mechanics and thermodynamics to fractal porous media which are specified by a mass (or spatial) fractal dimension D, a surface fractal dimension d, and a resolution lengthscale R. The focus is on pre-fractal media (i.e., those with lower and upper cut-offs) through a theory based on dimensional regularization, in which D is also the order of fractional integrals employed to state global balance laws. In effect, the global forms of governing equations may be cast in forms involving conventional (integer-order) integrals, while the local forms are expressed through partial differential equations with derivatives of integer order but containing coefficients involving D, d and R. Here we first generalize the principles of virtual work, virtual displacement and virtual stresses, which in turn allow us to extend the minimum energy theorems of elasticity theory. Next, we generalize the extremum principles of elasto-plastic and rigid-plastic bodies. In all the cases, the derived relations depend explicitly on D, d and R, and, upon setting $D = 3$ and $d = 2$, they reduce to conventional forms of governing equations for continuous media with Euclidean geometries.

1 Background and motivation

The continuum property is desired in providing a passage from a random heterogeneous microstructure to a homogenizing continuum. While a number of methods have been developed over the past few decades to justify this passage in the setting of materials having Euclidean geometries (e.g. [1,2]), in the case of fractal (i.e., almost everywhere non-differentiable) media, novel methods outside classical continuum mechanics have to be employed.

Back in 1975, Benoît Mandelbrot coined the term “fractal” to refer to an object that is “broken” or “fractured” in space and/or time [3]. In general, a fractal object can be subdivided in parts, each of which is in a deterministic or stochastic sense a reduced-size copy of the whole; this is the famous self-similarity property (1). A fractal also has these features: (2) Fine structure at arbitrarily small scales; (3) too irregular to be easily described in traditional Euclidean geometric language; (4) Hausdorff dimension which is greater than its topological dimension; (5) a simple and recursive definition.

It follows that “mathematical fractals” appear similar at all levels of magnification, and, roughly speaking, they are infinitely complex [4]. [There are various exceptions to the list of properties (1–5) above, e.g. (a) not all self-similar objects are fractals—for example, the real line (a straight Euclidean line) is formally self-similar but fails to have other fractal characteristics. (b) Space-filling curves (e.g. the Hilbert curve) do not satisfy (4)]. Now, fractals in space, as opposed to those in time (signals, processes), approximate spatial fractality of many natural and man-made objects to a degree; examples are coastlines, porous media, cracks, turbulent flows,

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clouds, mountains, lightning bolts, brains, snow flakes, melting ice (and other systems at phase transitions). However, such media are fractal-like only over some finite range of length scales defined by lower and upper cutoffs present. These are so-called "pre-fractals" as opposed to mathematical fractals which have no cut-offs; at this point we refer to an interesting discussion by Avnir et al. [5] and Mandelbrot et al. [6].

Much research has been carried out on pre-fractals primarily in condensed matter physics (e.g. [7]), focusing on physical phenomena and properties for materials whose fractal-like geometry plays a key role. However, a field theory, an analogue of continuum physics and mechanics, has sorely been lacking. Some progress in that respect has recently been made by mathematicians [8,9], and only simple problems, like Laplace's or heat equation, on fractal (albeit non-random) sets begin to be tackled. This approach is very technical from the mathematical analysis standpoint and focuses on mathematical fractals rather than pre-fractals.

A very different step in the direction of a field theory and initial-boundary value problems has recently been taken by Tarasov [10–12]. He developed continuum-type equations of conservation of mass, linear and angular momenta, and energy for fractal porous media, and, on that basis investigated several problems in fluid mechanics and wave propagation. The mass in such media obeys a power law relation

$$m(R) = kR^{D}, \quad D < 3, \tag{1.1}$$

where R is a box size (or a sphere radius, effectively a lengthscale of measurement), D is a fractal dimension of mass, and k is a proportionality constant. The Eq. (1.1) implies that the conventional relation giving mass in a three-dimensional region W (of volume V and boundary ∂W)

$$m(W) = \int_{W} \rho(r)\, \mathrm{d}^{3}r \tag{1.2}$$

has to be generalized to

$$m_{3d}(W) = \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} \int_{W} \rho(r)\, |r - r_0|^{D-3}\, \mathrm{d}^{3}r. \tag{1.3}$$

The beauty and power of Tarasov's approach relies on a generalization of the Gauss theorem to fractal objects through fractional integrals in Euclidean space. That is, the fractal medium with a non-integer mass dimension D is described using a fractional integral of order D. This interpretation of the fractal (intrinsically discontinuous) medium as a continuum—in the vein of *dimensional regularization* of quantum mechanics [13]—a reformulation of the Gauss (or divergence) Theorem

$$\int_{\partial W} f_k n_k \mathrm{d}A_d = \int_{W} c_3^{-1}(D, R)\, \nabla_k \left(c_2(d, R)\, f_k\right) \mathrm{d}V_D, \tag{1.4}$$

where f_k is a vector field (in subscript notation) and

$$\mathrm{d}A_d = c_2(d, R)\, \mathrm{d}A_2, \quad \mathrm{d}V_D = c_3(D, R)\, \mathrm{d}V_3. \tag{1.5}$$

Here $\mathrm{d}A_2$ and $\mathrm{d}V_3$ are, respectively, the conventional infinitesimal elements of surface and volume in the Euclidean space. Hereinafter, we shall interchangeably use the symbolic ($\mathbf{u}$) and the subscript ($u_{i\ldots}$) notations for tensors, as the need arises; also an overdot will mean $\mathrm{d}/\mathrm{d}t$.

Note that the left-hand side in (1.4) is a fractional integral, equal to a conventional integral $\int_{\partial W} c_2(d, R)$ $f v_k n_k \mathrm{d}A_2$, while the right-hand side is a fractional integral, equal to a conventional integral $\int_W \mathrm{div}(c_2(d, R)$ $f v_k)\mathrm{d}V_3$. Thus, we can rewrite (1.4) as

$$\int_{\partial W} c_2(d, R)\, f_k n_k \mathrm{d}A_2 = \int_{W} \nabla_k \left(c_2(d, R)\, f_k\right) \mathrm{d}V_3, \tag{1.6}$$

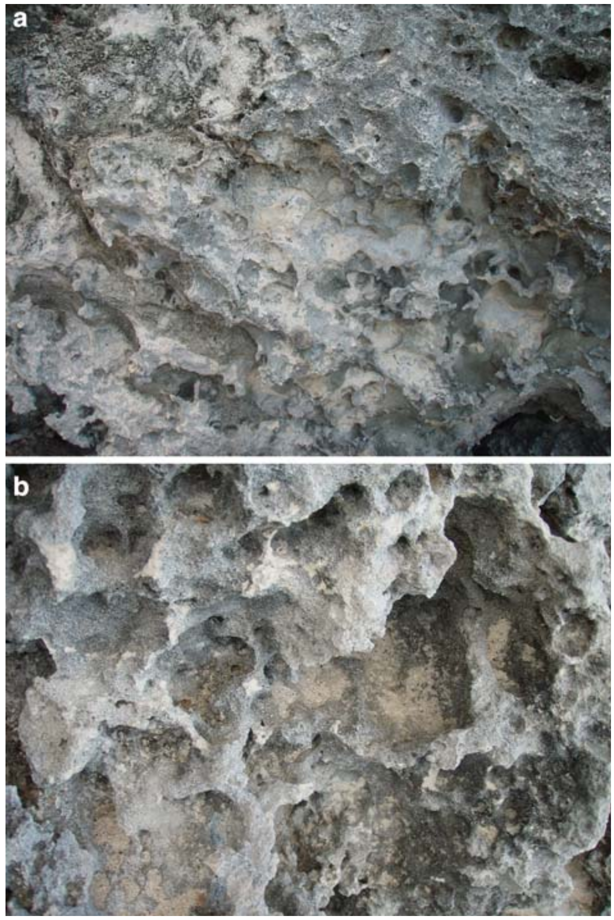

Fig. 1 A porous rock at two resolutions, $R = 1m$ (**a**) and $R = 0.1m$ (**b**), on the sea shore in Tulum, Mexico, representing an example of a pre-fractal

and, in fact, extend this theorem to the setting with a jump $[f_k]$ on a surface S across W

$$\int_{\partial W} c_2(d, R)\, f_k n_k \mathrm{d}A_2 = \int_{W-S} \nabla_k (c_2(d, R)\, f_k)\, \mathrm{d}V_3 + \int_{S} c_2(d, R)\, [f_k]\, n_k \mathrm{d}A_2. \tag{1.7}$$

Another advantage of Tarasov's approach is that it admits upper and lower cut-offs of fractal scaling, so that one effectively deals with a physical "pre-fractal" (see Fig. 1) rather than a purely mathematical fractal. It is in that sense that fractals are meant here. In principle, one can then map a mechanics problem of a fractal [which is described by its mass (D) and surface (d) fractal dimensions plus the spatial resolution (R)] onto a problem in the Euclidean space in which this fractal is embedded, while having to deal with coefficients explicitly involving D, d and R. Clearly, this has very interesting ramifications for formulating continuum-type mechanics of fractal media, which need to be further explored. The great promise stems from the fact that the conventional requirement of continuum mechanics of the separation of scales can be dropped, yet the partial differential equations may still be employed.

In [14–17] we have obtained generalizations of the Clausius–Duhem inequality, the linear thermoelasticity, the Maxwell–Betti reciprocity, the Hill condition and energy principles, the energy release rate method in fracture mechanics, the mean equations of turbulence, as well as the Stokes' and Reynolds' (transport) theorems. In the present paper we formulate the extremum and variational principles of elasticity and plasticity for fractal porous materials. In particular, we derive relations which depend explicitly on D, d and R, and which, upon setting $D = 3$ and $d = 2$, reduce to conventional (well known) forms of governing equations for continuous media with Euclidean geometries.

2 Balance equations in the setting of fractal media

The form (1.4) has led Tarasov to introduce the following operators generalizing conventional derivatives:

$$\nabla_k^D f = c_3^{-1}(D,R)\frac{\partial}{\partial x_k}\left[c_2(d,R)f\right] \equiv c_3^{-1}(D,R)\nabla_k\left[c_2(d,R)f\right]$$
$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_D f = \frac{\partial f}{\partial t} + c(D,d,R)\,v_k\frac{\partial f}{\partial x_k}, \tag{2.1}$$

where

$$c(D,d,R) = |\mathbf{R}|^{d+1-D}\frac{2^{D-d-1}\Gamma(D/2)}{\Gamma(3/2)\Gamma(d/2)} = c_3^{-1}(D,R)\,c_2(d,R)$$
$$c_2(d,R) = |\mathbf{R}|^{d-2}\frac{2^{2-d}}{\Gamma(d/2)} \tag{2.2}$$
$$c_3(D,R) = |\mathbf{R}|^{D-3}\frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)}.$$

Among the principal features of the theory thus proposed are:

(i) The volume $\mathrm{d}V_3$ is not really infinitesimal but an upper length cutoff for the fractal structure. The lower one is given by whatever molecular scale in a specific problem. Thus, the theory is suited for physical fractals—sometimes called *pre-fractals*—as opposed to mathematical fractals without any cutoffs; recall Fig. 1.

(ii) The theory involves fractional integrals but conventional derivatives plus the c_3 and c_2 coefficients, with the order of integrals being directly given by fractal dimensions of volumes (D) and surfaces (d). Having conventional derivatives makes the present theory easier to deal with than those with fractional derivatives.

(iii) In view of (ii), the theory is limited to homogeneous, isotropic fractal media.

(iv) The equations governing problems in 1-D cannot be consistently obtained from the equations governing problems in 3-D.

Points (iii) and (iv) represent drawbacks stemming from the fact that c_2 and c_3 are based on the Riesz measure. That drawback can be removed by introducing a product measure instead, thereby ensuring that the mechanical approach to continuum mechanics is consistent with the energetic approach. To this end, note that, while the mass distribution in conventional continuum mechanics [18] is

$$\mathrm{d}\mu(\mathbf{x}) = \rho(\mathbf{x})\mathrm{d}V_3, \tag{2.3}$$

with $\rho(x)$ being the mass density and $\mathrm{d}V_3$ the Lebesgue measure in $\mathbb{R}^3$, the product measure we introduce [19] is

$$\mathrm{d}\mu_k(x_k) = \rho(\mathbf{x})c_1(\alpha_k, x_k)\mathrm{d}x_k, \quad k = 1, 2, 3. \tag{2.4}$$

Thus, whereas (2.3) applies to a non-fractal mass distribution $M \sim x^3$, (2.4) applies to a fractal mass distribution $M \sim x^{\alpha_1}x^{\alpha_2}x^{\alpha_3}$, the total fractal dimension being $D = \alpha_1 + \alpha_2 + \alpha_3$. Adopting a Riemann–Liouville fractional integral, we have

$$c_1^{(k)} = \frac{|x_k|^{\alpha_k - 1}}{\Gamma(\alpha_k)}, \quad k = 1, 2, 3, \tag{2.5}$$

so as to replace (2.2) by

$$c_2^{(k)} = c_1^{(i)}c_1^{(j)} = \frac{|x_i|^{\alpha_i-1}|x_j|^{\alpha_j-1}}{\Gamma(\alpha_i)\Gamma(\alpha_j)}, \quad i, j \neq k, \quad i \neq j$$
$$c_3 = c_1^{(i)}c_1^{(j)}c_1^{(k)} = \frac{|x_1|^{\alpha_i-1}|x_2|^{\alpha_j-1}|x_3|^{\alpha_k-1}}{\Gamma(\alpha_i)\Gamma(\alpha_j)\Gamma(\alpha_k)}. \tag{2.6}$$

Now, whether one works with c_2 and c_3 based on the Riesz measure (more suited to fluid mechanics) or with c_2 and c_3 based on the product measure (more suited to solid mechanics), one has these modified balance equations of fractal media [10–12,14]

- the fractional equation of continuity:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_D \rho = -\rho \nabla_k^D v_k, \tag{2.7}$$

- the fractional equation of balance of density of momentum:

$$\rho\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_D v_k = F_k + \nabla_l^D \sigma_{kl} \quad \text{or} \quad \rho\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_D v_k = \rho f_k + \nabla_l^D \sigma_{kl}, \tag{2.8}$$

- the fractional equation of balance of density of energy:

$$\rho\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_D u = c\,(D, d, R)\,\sigma_{kl} v_{k,l} - \nabla_k^D q_k, \tag{2.9}$$

- the Clausius–Duhem inequality:

$$0 \le T\rho\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_D s^{(i)} = \sigma_{ij}^{(d)}\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_D u_{(i}\right]_{,j)} + \beta_{ij}^{(d)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_D \alpha_{ij} - c\,(D, d, R)\,\frac{T_{,k}\, q_k}{T}. \tag{2.10}$$

In the above σ_{kl} is the Cauchy stress (symmetric according to the balance of angular momentum, employed just like in non-fractal media), while $\sigma_{ij}^{(d)}$ and $\beta_{ij}^{(d)}$ are the dissipative stresses, while α_{ij} are the internal parameters in the vein of thermomechanics with internal variables [20,21].

Constitutive laws of fractal media now follow from the above, whereby one can distinguish two fundamental cases: compound or complex thermodynamical processes. The key thing to observe is that, when the dissipation function is taken as a functional in derivatives $(\mathrm{d}/\mathrm{d}t)_D\varepsilon_{ij}$ and $(\mathrm{d}/\mathrm{d}t)_D\alpha_{ij}$, a number of key relations of Ziegler's thermomechanics theory—such as laws governing complex and compound processes and the associated Onsager–Casimir relations and Legendre transformations—carry over to fractal media.

However, it follows from a careful consideration of the Reynolds (transport) Theorem for fractal media [16] that $(\mathrm{d}/\mathrm{d}t)_D$ in (2.1.2) should actually be replaced by the conventional material derivative

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + v_k \frac{\partial f}{\partial x_k}. \tag{2.11}$$

As a result, all the previous results simplify somewhat. However, whichever interpretation of the time derivative is employed, in a non-fractal medium specified by $D = 3$ and $d = 2$, all three c coefficients in (2.2) become equal to 1, whereby one recovers conventional forms of transport and balance equations of continuum mechanics. All the results reported below possess the same feature.

3 Extremum and variational principles in elasticity

Define a *statically admissible field* as a tensor function $\sigma_{ij}(\mathbf{x})$, such that $\sigma_{ij} = \sigma_{ji}$ and $(F_k = \rho f_k)$

$$F_k + \nabla_l^D \sigma_{kl} = 0 \tag{3.1}$$

in W and the boundary conditions

$$\sigma_{kl} n_l = t_k \tag{3.2}$$

on ∂W_t.

Define a *kinematically admissible displacement field* as a vector function $\mathbf{u}(\mathbf{x})$ satisfying the boundary conditions

$$u_i = f_i \tag{3.3}$$

on ∂W_u.

Principle of virtual work: "The virtual work of the internal forces equals the virtual work of the external forces". Let $\boldsymbol{\sigma}(\mathbf{x})$ be a statically admissible stress field, $\mathbf{u}(\mathbf{x})$ a kinematically admissible displacement field. Define $\varepsilon_{ij}(\mathbf{u}) = u_{(i,j)}$. Then

$$\int_W c(D,d,R)\sigma_{ij}\varepsilon_{ij}\mathrm{d}V_D = \int_W F_i u_i \mathrm{d}V_D + \int_{\partial W_t} t_i u_i \mathrm{d}S_d + \int_{\partial W_t} \sigma_{ij} n_j f_i \mathrm{d}S_d. \tag{3.4}$$

The proof follows by substitution from the fractional equation of static equilibrium and boundary conditions after integrating by parts, and using the Gauss theorem. In terms of conventional integrals:

$$\int_W c_2(d,R)\sigma_{ij}\varepsilon_{ij}\mathrm{d}V_3 = \int_W c_3(d,R)F_i u_i \mathrm{d}V_3 + \int_{\partial W_t} c_2(d,R) t_i u_i \mathrm{d}S_2 + \int_{\partial W_t} c_2(d,R)\sigma_{ij} n_j f_i \mathrm{d}S_2. \tag{3.5}$$

In the following we apply the same techniques (i.e., using the rules of calculus of Sect. 2) to generalize other well-known principles.

Principle of virtual displacement: Since $\mathbf{u}^0$ is the classical solution, it is also a kinematically admissible displacement field. Let σ^* be a statically admissible stress field, and substitute it into the principle of virtual work. Also, substitute $\mathbf{u}^0$ into that principle. Separately, substitute $\mathbf{u}^0 + \delta\mathbf{u}$ into an arbitrary displacement field, which is a kinematically admissible displacement field, and subtract the resultant equations to obtain:

$$\int_W c(D,d,R)\sigma_{ij}\delta\varepsilon_{ij}\mathrm{d}V_D = \int_W F_i \delta u_i \mathrm{d}V_D + \int_{\partial W_t} t_i \delta u_i \mathrm{d}S_d, \tag{3.6}$$

where $\delta\varepsilon_{ij} = \varepsilon_{ij}(\delta\mathbf{u})$, being the virtual displacement (or variation of $\mathbf{u}$). It follows from (3.3) that $\delta\mathbf{u} = \mathbf{0}$ on ∂W_u, but is an arbitrary (though sufficiently smooth) vector field elsewhere. In terms of conventional integrals:

$$\int_W c_2(d,R)\sigma_{ij}\delta\varepsilon_{ij}\mathrm{d}V_3 = \int_W c_2(d,R)F_i \delta u_i \mathrm{d}V_3 + \int_{\partial W_t} c_2(d,R) t_i \delta u_i \mathrm{d}S_2. \tag{3.7}$$

Principle of virtual stresses: Substitute the actual displacement field $\mathbf{u}^0$ into the principle of virtual work. Then, subtract the resulting equation with the actual stress field $\boldsymbol{\sigma}^0$ from that with a statically admissible field $\boldsymbol{\sigma}^0 + \delta\boldsymbol{\sigma}^0$. We then obtain:

$$\int_W c(D,d,R)\delta\sigma_{ij}\varepsilon_{ij}(\mathbf{u}^0)\mathrm{d}V_D = \int_{\partial W_u} f_i \delta\sigma_{ij} n_j \mathrm{d}S_d, \tag{3.8}$$

with $\delta\sigma_{ij}$ being the virtual stress (or variation of $\boldsymbol{\sigma}$). It follows from (3.1) that $\delta\mathbf{u} = \mathbf{0}$ on ∂W_u, but is an arbitrary (though sufficiently smooth) vector field elsewhere. In terms of conventional integrals:

$$\int_W c_2(d,R)\delta\sigma_{ij}\varepsilon_{ij}(\mathbf{u}^0)\mathrm{d}V_3 = \int_{\partial W_u} c_2(d,R) f_i \delta\sigma_{ij} n_j \mathrm{d}S_2. \tag{3.9}$$

Principle of minimum potential energy: The solution $\mathbf{u}^0$ of the mixed problem of classical (linear) elasticity theory gives for the functional of potential energy

$$U(\mathbf{u}) = \frac{1}{2}\int_W c(D,d,R)C_{ijkl}\varepsilon_{ij}(\mathbf{u})\varepsilon_{kl}(\mathbf{u})\mathrm{d}V_D - \int_W c(D,d,R)F_i u_i \mathrm{d}V_D - \int_{\partial W_t} t_i u_i \mathrm{d}S_d \tag{3.10}$$

its minimum value over the set of kinematically admissible displacement fields. In terms of conventional integrals:

$$U(\mathbf{u}) = \frac{1}{2}\int_W c_2(d,R)C_{ijkl}\varepsilon_{ij}(\mathbf{u})\varepsilon_{kl}(\mathbf{u})\mathrm{d}V_3 - \int_W c_3(d,R)F_i u_i \mathrm{d}V_3 - \int_{\partial W_t} c_2(d,R) t_i u_i \mathrm{d}S_2. \tag{3.11}$$

We have recently shown [17] that another way of writing that principle is

$$\int\limits_{\partial W_t} c_2(d,R)u_i^* t_i \mathrm{d}A_2 - \frac{1}{2}\int\limits_W c_2(d,R)\sigma_{ij}^*\varepsilon_{ij}^* \mathrm{d}V_3 < \int\limits_{\partial W_t} c_2(d,R)u_i t_i \mathrm{d}A_2 - \frac{1}{2}\int\limits_W c_2(d,R)\sigma_{ij}\varepsilon_{ij}\mathrm{d}V_3, \quad (3.12)$$

which means that the expression on the left hand side of (3.12) takes an absolute maximum value in the actual state. As before, * denotes kinematically admissible fields.

Principle of minimum complementary energy: If the actual stress field $\boldsymbol{\sigma}^0$ of the mixed problem of classical (linear) elasticity theory is statically admissible, it gives the functional of potential energy

$$U^*(\mathbf{u}) = \frac{1}{2}\int\limits_W c(D,d,R)S_{ijkl}\sigma_{ij}\sigma_{kl}\mathrm{d}V_D - \int\limits_{\partial W_u} f_i\sigma_{ij}n_j\mathrm{d}S_d \quad (3.13)$$

its minimum value over the set of all statically admissible stress fields. In terms of conventional integrals:

$$U^*(\mathbf{u}) = \frac{1}{2}\int\limits_W c_2(d,R)S_{ijkl}\sigma_{ij}\sigma_{kl}\mathrm{d}V_3 - \int\limits_{\partial W_u} c_2(d,R)f_i\sigma_{ij}n_j\mathrm{d}S_2 \quad (3.14)$$

We have recently shown [17] that another way of writing that principle is

$$\frac{1}{2}\int\limits_W c_2(d,R)\sigma_{ij}^*\varepsilon_{ij}^*\mathrm{d}V_3 - \int\limits_{\partial W_u} c_2(d,R)t_i^* u_i\mathrm{d}S_2 > \frac{1}{2}\int\limits_{\partial W_t} c_2(d,R)t_i u_i\mathrm{d}S_2 - \frac{1}{2}\int\limits_{\partial W_u} c_2(d,R)t_i u_i\mathrm{d}S_2, \quad (3.15)$$

where * denotes statically admissible fields. That is, the expression on the left-hand side of (3.15) takes an absolute minimum value in the actual state.

All the above relations imply that one can apply the extremum principles of elasticity to fractal bodies, provided extra information is taken into account through D, d and R.

4 Extremum principles in elasto-plasticity

Consider a *statically admissible field* denoted by *. We can then write

$$\int\limits_W c(D,d,R)(\mathrm{d}\sigma_{ij}^* - \mathrm{d}\sigma_{ij})\mathrm{d}\varepsilon_{ij}\,\mathrm{d}V_D = \int\limits_{\partial W}\left(\mathrm{d}t_i^* - \mathrm{d}t_i\right)\mathrm{d}u_i\,\mathrm{d}S_d \quad (4.1)$$

in view of the fractional equation of static equilibrium for a body without a body force field.

Now, at every point in the pre-fractal elastic body, just like in a non-fractal elastic body, this inequality holds

$$\frac{1}{2}(\mathrm{d}\sigma_{ij}^*\mathrm{d}\varepsilon_{ij} - \mathrm{d}\sigma_{ij}\mathrm{d}\varepsilon_{ij}) > (\mathrm{d}\sigma_{ij}^* - \mathrm{d}\sigma_{ij})\mathrm{d}\varepsilon_{ij}. \quad (4.2)$$

unless $\mathrm{d}\sigma_{ij}^{*\prime} = \mathrm{d}\sigma_{ij}^{\prime}$. Hence

$$\frac{1}{2}\int\limits_W(\mathrm{d}\sigma_{ij}^*\mathrm{d}\varepsilon_{ij} - \mathrm{d}\sigma_{ij}\mathrm{d}\varepsilon_{ij})\,\mathrm{d}V_D > \int\limits_{\partial W_t}(\mathrm{d}\sigma_{ij}^* - \mathrm{d}\sigma_{ij})\mathrm{d}\varepsilon_{ij}\,\mathrm{d}S_d. \quad (4.3)$$

Thus, (5.1) is written as

$$\begin{aligned}\frac{1}{2}\int\limits_W c(D,d,R)\mathrm{d}\sigma_{ij}^*\mathrm{d}\varepsilon_{ij}\,\mathrm{d}V_D - \int\limits_{\partial W}\mathrm{d}t_i^*\mathrm{d}u_i\,\mathrm{d}S_d &> \frac{1}{2}\int\limits_W c(D,d,R)\mathrm{d}\sigma_{ij}\mathrm{d}\varepsilon_{ij}\,\mathrm{d}V_D - \int\limits_{\partial W_u}\mathrm{d}t_i\mathrm{d}u_i\,\mathrm{d}S_d \\ &= \frac{1}{2}\int\limits_{\partial W_t} c(D,d,R)\mathrm{d}t_i\mathrm{d}u_{ij}\,\mathrm{d}S_d - \int\limits_{\partial W_u}\mathrm{d}t_i\mathrm{d}u_i\,\mathrm{d}S_d \quad (4.4)\end{aligned}$$

Consider a *kinematically admissible field* denoted by *. Just like in case of (4.1), we can then prove

$$\int_W c(D, d, R)(\mathrm{d}\varepsilon_{ij}^* - \mathrm{d}\varepsilon_{ij})\mathrm{d}\sigma_{ij}\, \mathrm{d}V_D = \int_{\partial W} \left(\mathrm{d}u_i^* - \mathrm{d}u_i\right)\mathrm{d}t_i\, \mathrm{d}S_d. \tag{4.5}$$

Now, at every point in the pre-fractal elastic body, just like in a non-fractal elastic body, this inequality holds:

$$\frac{1}{2}(\mathrm{d}\sigma_{ij}^*\mathrm{d}\varepsilon_{ij}^* - \mathrm{d}\sigma_{ij}\mathrm{d}\varepsilon_{ij}) > (\mathrm{d}\varepsilon_{ij}^* - \mathrm{d}\varepsilon_{ij})\mathrm{d}\sigma_{ij} \tag{4.6}$$

unless $\sigma_{ij}^{*\prime} = \sigma_{ij}^{\prime}$. Hence,

$$\int_W \frac{1}{2}(\mathrm{d}\sigma_{ij}^*\mathrm{d}\varepsilon_{ij}^* - \mathrm{d}\sigma_{ij}\mathrm{d}\varepsilon_{ij})\mathrm{d}V_D > \int_{\partial W} (\mathrm{d}\varepsilon_{ij}^* - \mathrm{d}\varepsilon_{ij})\mathrm{d}\sigma_{ij}\mathrm{d}S_d. \tag{4.7}$$

This leads to

$$\begin{aligned}\int_{\partial W_t} \mathrm{d}u_i^*\mathrm{d}t_i\, \mathrm{d}S_d - \frac{1}{2}\int_W c(D, d, R)\mathrm{d}\sigma_{ij}^*\mathrm{d}\varepsilon_{ij}^*\, \mathrm{d}V_D &< \int_{\partial W_t} \mathrm{d}u_i\mathrm{d}t_i\, \mathrm{d}S_d - \frac{1}{2}\int_W c(D, d, R)\mathrm{d}\sigma_{ij}\mathrm{d}\varepsilon_{ij}\, \mathrm{d}V_D \\ &= \frac{1}{2}\int_{\partial W_t} c(D, d, R)\mathrm{d}t_i\mathrm{d}u_{ij}\, \mathrm{d}S_d - \int_{\partial W_u} \mathrm{d}t_i\mathrm{d}u_i\, \mathrm{d}S_d,\end{aligned} \tag{4.8}$$

which, in view of (1.5), can be written in terms of conventional integrals in the Euclidean space

$$\begin{aligned}\int_{\partial W_t} \mathrm{d}u_i^*\mathrm{d}t_i\, c_2\mathrm{d}S_2 - \frac{1}{2}\int_W c(D, d, R)\mathrm{d}\sigma_{ij}^*\mathrm{d}\varepsilon_{ij}^*\, c_3\mathrm{d}V_3 &< \int_{\partial W_t} \mathrm{d}u_i\mathrm{d}t_i\, c_2\mathrm{d}S_d - \frac{1}{2}\int_W c(D, d, R)\mathrm{d}\sigma_{ij}\mathrm{d}\varepsilon_{ij}\, c_3\mathrm{d}V_D \\ &= \frac{1}{2}\int_{\partial W_t} c(D, d, R)\mathrm{d}t_i\mathrm{d}u_{ij}\, c_2\mathrm{d}S_d - \int_{\partial W_u} \mathrm{d}t_i\mathrm{d}u_i\, c_2\mathrm{d}S_d.\end{aligned} \tag{4.9}$$

Thus, similar to the case of elastic bodies, one can apply the extremum principles of elasto-plasticity to fractal bodies, provided extra information is taken into account through D, d and R.

We end this section with an upper bound theorem allowing for discontinuities in the velocity field, according to [22]. It is formulated in the setting of a multi-phase elastic–plastic hardening material with perfect bonding between the phases $p = 1, \ldots, p_{\mathrm{tot}}$. The body W is described by an associated flow rule

$$\begin{aligned}\mathrm{d}\varepsilon_{ij}^{\prime} &= \frac{\mathrm{d}\sigma_{ij}^{\prime}}{2G_p} + \lambda\frac{\partial f}{\partial\sigma_{ij}}\mathrm{d}f_p \quad \text{whenever } f_p = c_p \quad \text{and} \quad \mathrm{d}f \geq 0, \\ \mathrm{d}\varepsilon_{ij}^{\prime} &= \frac{\mathrm{d}\sigma_{ij}^{\prime}}{2G_p} \quad \text{whenever } f_p < c_p, \\ \mathrm{d}\varepsilon &= \frac{1-2\nu_p}{2G_p(1+\nu_p)}\mathrm{d}\sigma \quad \text{everywhere } (\mathrm{d}\varepsilon = \mathrm{d}\varepsilon_{ii}/3,\ \mathrm{d}\sigma = \mathrm{d}\sigma_{ii}/3).\end{aligned} \tag{4.10}$$

Here G_p is the shear modulus, ν_p the Poisson's ratio, and c_p the yield limit. Next, we consider an arbitrary, kinematically admissible velocity field v^* of the body W. If σ_{ij}^* is the stress field associated with v_i^* by (4.10.1) and also satisfying (4.10.2), the basic energy balance equation can now be written as

$$\int_{\partial W} c_2(d, R)\, t_i v_i^*\mathrm{d}S_d = \int_W c(D, d, R)\sigma_{ij}d_{ij}^*\mathrm{d}V_D + \int_{S[v^*]} c_2(d, R)\,\tau_Y\left|\left[v^*\right]\right|\mathrm{d}S_d, \tag{4.11}$$

where $S^{[v^*]} = \cup_{m=1}^{M} S_m^{[v^*]}$ is the set of internal surfaces of discontinuity in v^*. In view of the inequality

$$\sigma_{ij} d_{ij}^* \leq \sigma_{ij}^* d_{ij}^*, \tag{4.12}$$

we find

$$\int\limits_{\partial W} c_2\left(d, R\right) t_i v_i^* \mathrm{d}S_d \leq \int\limits_{W} c(D, d, R) \sigma_{ij}^* d_{ij}^* \mathrm{d}V_D + \tau_Y \int\limits_{S^{[v^*]}} c_2\left(d, R\right) \left|\left[v^*\right]\right| \mathrm{d}S_d. \tag{4.13}$$

5 Extremum principles in rigid-plasticity

Consider a *statically admissible field* denoted by *. We can then write

$$\int\limits_{W} c(D, d, R)(\mathrm{d}\sigma_{ij}^* - \mathrm{d}\sigma_{ij})\mathrm{d}\varepsilon_{ij}\ \mathrm{d}V_D = \int\limits_{\partial W_u} \left(\mathrm{d}t_i^* - \mathrm{d}t_i\right) \mathrm{d}u_i\ \mathrm{d}S_d \tag{5.1}$$

in view of the fractional equation of static equilibrium for a body without a body force field.

Now, at every point in the pre-fractal elastic body, just like in a non-fractal elastic body, this inequality holds:

$$(\sigma_{ij}^* - \sigma_{ij})\mathrm{d}\varepsilon_{ij} < 0 \tag{5.2}$$

unless $\sigma_{ij}^{*\prime} = \sigma_{ij}'$. Thus, (5.1) is written as

$$\int\limits_{\partial W_u} t_i^* \mathrm{d}u_i\ \mathrm{d}S_d < \int\limits_{\partial W_u} t_i \mathrm{d}u_i\ \mathrm{d}S_d \tag{5.3}$$

or, equivalently,

$$\int\limits_{\partial W_u} t_i^* \mathrm{d}u_i c_2(d, R)\ \mathrm{d}S_2 < \int\limits_{\partial W_u} t_i \mathrm{d}u_i c_2(d, R)\ \mathrm{d}S_2. \tag{5.4}$$

Consider a *kinematically admissible field* denoted by *. Just like in case of (5.1), we can then prove

$$\int\limits_{\partial W_t} \left(\mathrm{d}u_i^* - \mathrm{d}u_i\right) t_i\ \mathrm{d}S_d = \int\limits_{W} c(D, d, R) \left(\mathrm{d}\varepsilon_{ij}^* - \mathrm{d}\varepsilon_{ij}\right) \sigma_{ij}\ \mathrm{d}V_D. \tag{5.5}$$

Now, at every point in the pre-fractal elastic body, just like in a non-fractal elastic body, this inequality holds:

$$\sigma_{ij}(\mathrm{d}\varepsilon_{ij}^* - \mathrm{d}\varepsilon_{ij}) < |\sigma_{ij}'| \left(|\mathrm{d}\varepsilon_{ij}^*| - |\mathrm{d}\varepsilon_{ij}|\right), \tag{5.6}$$

unless $\sigma_{ij}^{*\prime} = \sigma_{ij}'$. This leads to

$$\begin{aligned} &\int\limits_{W} c(D, d, R)|\sigma_{ij}'||\mathrm{d}\varepsilon_{ij}^*|\ \mathrm{d}V_D - \int\limits_{\partial W_t} t_i \mathrm{d}u_i^*\ \mathrm{d}S_d \\ &\quad > \int\limits_{W} c(D, d, R)|\sigma_{ij}'||\mathrm{d}\varepsilon_{ij}|\ \mathrm{d}V_D - \int\limits_{\partial W_t} t_i \mathrm{d}u_i\ \mathrm{d}S_d = \int\limits_{\partial W_u} t_i \mathrm{d}u_i\ \mathrm{d}S_d, \end{aligned} \tag{5.7}$$

which, in view of (1.5), can be written in terms of conventional integrals in the Euclidean space

$$\int\limits_{W} c_2(d, R)|\sigma_{ij}'||\mathrm{d}\varepsilon_{ij}^*|\ \mathrm{d}V_3 - \int\limits_{\partial W_t} c_2(d, R) t_i \mathrm{d}u_i^*\ \mathrm{d}S_2 > \int\limits_{\partial W_u} c_2(d, R) t_i \mathrm{d}u_i\ \mathrm{d}S_2. \tag{5.8}$$

Thus, similar to the case of elastic bodies, one can apply the extremum principles of plasticity to fractal bodies, provided extra information is taken into account through D, d and R.

6 Closure

In this paper, we continue the recently begun extension of continuum mechanics and thermodynamics to fractal porous media which are specified by a mass (or spatial) fractal dimension D, a surface fractal dimension d, and a resolution lengthscale R. The focus is on pre-fractal media (i.e., those with lower and upper cut-offs) through a theory based on dimensional regularization, in which D is also the order of fractional integrals employed to state global balance laws. In effect, the global forms of governing equations may be cast in forms involving conventional (integer-order) integrals, while the local forms are expressed through partial differential equations with derivatives of integer order but containing coefficients involving D, d and R. Here we first generalize the principles of virtual work, virtual displacements and virtual stresses, which in turn allow us to extend the minimum energy theorems of elasticity theory. Next, we generalize the extremum principles of elasto-plastic and rigid-plastic bodies. A similar procedure can also be applied in the case of, say, linear viscoelasticity theory, non-Fourier type heat conduction or hyperbolic thermoelasticity [23]. In all the principles treated here, the derived relations depend explicitly on D, d and R, and, upon setting $D = 3$ and $d = 2$, reduce to conventional forms of governing equations for continuous media with Euclidean geometries.

Acknowledgment This research was made possible by the support from the NSF under grant CMMI-0833070.

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