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Short communication

A derivation of the Maxwell–Cattaneo equation from the free energy and dissipation potentials

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A B S T R A C T

A thermodynamic derivation is presented of the Maxwell–Cattaneo equation involving a material time, instead of a partial time, derivative of heat flux. The Ansatz is given by the functionals of free energy and dissipation potentials, relying on an extended state space and a representation theory of Edelen.

Recently, Christov and Jordan [1] have shown that the Maxwell–Cattaneo equation governing the propagation of second sound should involve a material time derivative of heat flux ($\dot{q} \equiv Dq/Dt$) instead of a partial time derivative ($\partial q/\partial t$). That is, supposing we look at a one-dimensional (1-D) setting, this equation should read

$$q + t_0 \frac{Dq}{Dt} = -k\theta_{,x}, \tag{1}$$

where θ is the absolute temperature, t_0 is the relaxation time, and k is the thermal conductivity.

A question arises: Can (1) be justified by thermodynamics directly? In particular, can it be derived from two functionals playing roles of potentials: the free energy ψ and the dissipation function ϕ? It turns out that this cannot be done using the thermodynamic orthogonality within the framework of thermodynamics with internal variables (TIV) [2], even when the thermodynamic state space is extended to include the heat flux or another quantity (e.g. the temperature rate). It is understood [3] that an extension of that type is needed, but, to the best of our knowledge, a derivation has not yet been published. Interestingly, while extended thermodynamics readily involves broader state spaces than other theories, the equation we typically see there (e.g. [4]) involves a partial derivative of q:

$$q + t_0 \frac{\partial q}{\partial t} = -k\theta_{,x}. \tag{2}$$

Consistent with the said extension, we assume the (specific, per unit mass) internal energy u to be a function of the strains E_{ij}, the entropy η and the heat flux q_i

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$$u = e(E_{ij}, \eta, q_i) \tag{3}$$

and the (specific, per unit mass) dissipation ϕ to be a function of the heat flux and possibly its rate:

$$\phi = \phi(q_i, \dot{q}_i). \tag{4}$$

We are focusing on a rigid conductor, so that in the above we do not need to admit other fluxes or velocities.

Now, whether we use TIV or a rational thermodynamics approach, in 3-D we obtain the Clausis–Duhem inequality in the form

$$-\frac{q_i \theta_{,i}}{\theta} - \frac{\partial \psi}{\partial q_i} \dot{q}_i \geqslant 0. \tag{5}$$

Clearly, this may be written as

$$\mathbf{Y} \cdot \mathbf{v} \geqslant 0, \tag{6}$$

where

$$\mathbf{Y} = \left(-\frac{\nabla \theta}{\theta}, -\nabla_{\mathbf{q}} \psi \right) \tag{7}$$

is a vector of dissipative thermodynamic forces, and

$$\mathbf{v} = (\mathbf{q}, \dot{\mathbf{q}}) \tag{8}$$

is a vector of conjugate thermodynamic velocities. In (7) $\nabla_{\mathbf{q}}$ stands for the gradient in the space of heat flux $\mathbf{q}$. See also in [3, pp. 73–74].

A general procedure based on the representation theory due to Edelen [5–7] allows a derivation of the most general form of the constitutive relation either for $\mathbf{v}$ as a function of $\mathbf{Y}$ or for $\mathbf{Y}$ as a function of $\mathbf{v}$, subject to (6). If we are to pursue the second alternative, the following steps are involved: Assume $\mathbf{Y}$ to be a function of $\mathbf{v}$, and determine it as

$$\mathbf{Y} = \nabla_{\mathbf{v}} \phi + \mathbf{U}, \text{ or } \quad Y_i = \frac{\partial \phi}{\partial v_i} + U_i, \tag{9}$$

where the vector $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2)$ does not contribute to the entropy production

$$\mathbf{U} \cdot \mathbf{v} = 0, \tag{10}$$

while the dissipation function is

$$\phi = \int_0^1 \mathbf{v} \cdot \mathbf{Y}(\tau \mathbf{v}) d\tau \tag{11}$$

and $\mathbf{U}$ is uniquely determined, for given $\mathbf{Y}$, by

$$U_i = \int_0^1 \tau v_j \left[\frac{\partial Y_i(\tau \mathbf{v})}{\partial v_j} - \frac{\partial Y_j(\tau \mathbf{v})}{\partial v_i} \right] d\tau \quad \text{with} \quad \frac{\partial [Y_i(\tau \mathbf{v}) - U_i]}{\partial v_j} = \frac{\partial [Y_j(\tau \mathbf{v}) - U_j]}{\partial v_i}. \tag{12}$$

The symmetry relations $(12)_2$ reduce to the classical Onsager reciprocity conditions

$$\frac{\partial Y_i(\tau \mathbf{v})}{\partial v_j} = \frac{\partial Y_j(\tau \mathbf{v})}{\partial v_i} \tag{13}$$

if and only if $\mathbf{U} = \mathbf{0}$.

Focusing first on the 1-D case (with $\mathbf{v}$ becoming $(q, \dot{q})$), the simplest $\mathbf{U}$ satisfying (10) is

$$U_1 = \frac{\lambda t_0}{\theta} \dot{q}, \quad U_2 = -\frac{\lambda t_0}{\theta} q, \tag{14}$$

whereby the dissipation function is a quadratic form

$$\phi(\mathbf{v}) \equiv \phi(q, \dot{q}) = \frac{1}{2\theta} \lambda q^2 + \frac{1}{2} G \dot{q}^2. \tag{15}$$

On account of (9), we obtain

$$\begin{aligned} -\frac{\theta_{,x}}{\theta} &\equiv Y_1 = \frac{\lambda q}{\theta} + U_1 = \frac{\lambda}{\theta} q + \frac{\lambda t_0}{\theta} \dot{q}, \\ -\frac{\partial \psi}{\partial q} &\equiv Y_2 = G \dot{q} + U_2 = G \dot{q} - \frac{\lambda t_0}{\theta} q. \end{aligned} \tag{16}$$

Now, observe:

(i) The Eq. $(16)_1$ immediately yields the Maxwell–Cattaneo Eq. (1): $-k\theta_{,x} = q + t_0\dot{q}$, with $k = 1/\lambda$.

(ii) The Eq. $(16)_2$ is satisfied identically providing a quadratic form of the free energy $\psi(q) = t_0\lambda q^2/2\theta$ is adopted (just like in the thermoelasticity with one relaxation time [8]) with θ being set equal to θ_0, along with $G = 0$. Regarding the latter, note that $\dot{q}$ must still be kept as an argument of $\phi(q,\dot{q})$ in (4). Interestingly, the assumption $\theta \sim \theta_0$ is involved in $(16)_2$ but not in $(16)_1$.

(iii) Other expressions for $\mathbf{U}$ and $\phi(q,\dot{q})$ – but then leading to nonlinear heat conduction laws, and, therefore, to nonlinear field equations – can be explored henceforth. At this point, one may consider various forms of homogeneous and quasi-homogeneous functions in place of the term $\lambda q^2/2\theta$ in (15) [2]. Regarding the quadratic form above, note that the factor 1/2 could be incorporated into G in (15) because ϕ is not required to equal the entropy production rate.

(iv) Another derivation implying the Maxwell–Cattaneo law can be found in [6,7], although that approach involves a dissipation function as a functional of forces $[\phi = \phi(\mathbf{Y})]$ along with a split of thermodynamic velocities $\mathbf{v} = \nabla_{\mathbf{Y}}\phi + \mathbf{U}$ with the condition $\mathbf{U}\cdot\mathbf{Y} = 0$. Focusing on fluids, Edelen [8] looks for $\mathbf{v}$ as a function of $\mathbf{Y}$, adopts a dissipation function involving $\theta_{,i}/\theta$ and $\dot{\theta}$, and first derives a nonlinear partial differential equation for heat conduction, which in the third-order approximation becomes a telegraph equation essentially implying the validity of (1). However, our approach based on TIV and the representation theory can be applied not only to thermoelastic (e.g., thermoelasticity theories with one or two relaxation times) but also to thermo-inelastic solids with second sound effects.

(v) The derivation above may be generalized to 3-D with general anisotropy:

$$\mathbf{v} = (\mathbf{q}, \dot{\mathbf{q}}), \quad \mathbf{Y} = \left(-\frac{\nabla\theta}{\theta}, -\nabla_{\mathbf{q}}\psi\right), \quad \mathbf{U} = (\mathbf{U}_1, \mathbf{U}_2). \tag{17}$$

Now,

$$U_{1i} = \frac{\Lambda_{ij}t_0}{\theta}\dot{q}_j, \quad u_{2i} = -\frac{\lambda_{ij}t_0}{\theta}q_j \tag{18}$$

and the dissipation function is

$$\phi(q_i, \dot{q}_i) = \frac{1}{2\theta_0}\lambda_{ij}q_iq_j + \frac{1}{2}G_{ij}\dot{q}_i\dot{q}_j, \tag{19}$$

where λ_{ij} and Λ_{ij} are two positive definite second-rank tensors. The Eqs. (18) and (19) jointly lead to the Maxwell–Cattaneo equation in a generally anisotropic medium

$$q_i + T_{ij}\frac{Dq_j}{Dt} = -k_{ij}\frac{\partial\theta}{\partial x_j}, \tag{20}$$

where $k_{ij} = (\lambda_{ij})^{-1}$ is the Fourier-type thermal conductivity and $T_{ij} = k_{im}\Lambda_{mj}$.

Closure. The goal of our research was to determine whether one could set up the Maxwell–Cattaneo equation from the free energy and the dissipation function. That is, whether one could somehow extend the thermodynamic orthogonality theory of Ziegler. It turns out that this is not possible, but a more general interpretation after Edelen does allow it. Indeed, one can still start with two functionals (free energy and dissipation function) and in place of the thermodynamic orthogonality apply Eqs. (9)–(12). Then, a non-zero dissipation function is truly present – i.e. the first term in (15). Additionally, the second term has to be present (for the sake of extending the state space to include the heat flux rate) until it is set to zero in Eq. (16). As is well known, the temperature field is then governed by a telegraph equation

$$t_0\frac{\partial^2\theta}{\partial t^2} + \frac{\partial\theta}{\partial t} = \frac{k}{\rho c}\frac{\partial^2\theta}{\partial x^2}. \tag{21}$$

See also [1] for a more general equation governing heat conduction in a moving frame. To conclude, thermodynamic orthogonality in TIV has its limitations as is brought out not only be the hyperbolic heat conduction, but, say, by plasticity and viscoelastic fluids [9,10]. See also [11] for a new, yet related research direction on non-dissipative elastic solids. For a modern account of generalized dynamic thermoelasticity see [12].

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