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# Fractal materials, beams, and fracture mechanics

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**Abstract.** Continuing in the vein of a recently developed generalization of continuum thermomechanics, in this paper we extend fracture mechanics and beam mechanics to materials described by fractional integrals involving $D$, $d$ and $R$. By introducing a product measure instead of a Riesz measure, so as to ensure that the mechanical approach to continuum mechanics is consistent with the energetic approach, specific forms of continuum-type equations are derived. On this basis we study the energy aspects of fracture and, as an example, a Timoshenko beam made of a fractal material; the local form of elastodynamic equations of that beam is derived. In particular, we review the crack driving force $G$ stemming from the Griffith fracture criterion in fractal media, considering either dead-load or fixed-grip conditions and the effects of ensemble averaging over random fractal materials.

**Mathematics Subject Classification (2000).**

**Keywords.**

## 1. Introduction

The present study is not the first one on fracture of fractal materials - several different approaches were developed since the nineties, see e.g. [1-5] and references therein. However, this study follows the path developed over the past few years that extended the continuum mechanics to fractal porous media specified by a mass (or spatial) fractal dimension $D$, a surface fractal dimension $d$, and a resolution lengthscale $R$ [6-8]. Basic balance relations (conservation of mass, linear momentum and angular momentum) have been formulated for such media − be they fluid or solid − with the help of fractional integrals and generalized (albeit non-fractional) derivatives. Furthermore, adopting the framework of thermomechanics with internal variables, we have obtained generalizations of the Clausius–Duhem inequality, the linear thermoelasticity, the Maxwell-Betti reciprocity, the Hill(–Mandel) condition and energy principles, and the mean equations of turbulence in fractal porous media [9-12]. In very recent work we have developed the integral relations (Stokes and Reynolds Theorems) as well as extremum and variational principles for elastic and plastic media with fractal geometries, possibly involving jumps in field quantities [13]. In all the cases, upon setting $D = 3$ and $d = 2$, one recovers the conventional (very well known) forms of governing equations for continuous media with Euclidean geometries. When $D$ and $d$ are non-integer and known, one can enter them into the new formulas to determine the response of a given material body.

Our approach, like that of Tarasov [6-8], is based on a technique of theoretical physics: *dimensional regularization*, e.g. [14]. For a function $f(x)$ this is represented by

$$\int_W f(x) d^D x = \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_W f(x) x^{D-1} dx. \tag{1.1}$$

That is, we begin with a fractal object embedded in an Euclidean space whose spatial dimension is not 3 but rather some real number $D < 3$. The same is done with the surface of that object, which has its own fractal dimension $d$, with $d$ not necessarily equal to $D - 1$. The balance laws are then written in weak forms involving volume and surface integrals over the fractal object. Converting these to conventional integrals via dimensional regularization, results in strong (local) forms for fractal bodies [6-9]. The key role in that approach is played by the Green-Gauss theorem for fractal media

$$\int_{\partial W} f_k n_k dA_d = \int_W c_3^{-1}(D, R) \nabla_k (c_2(d, R) f_k) dV_D, \tag{1.2}$$

where $f_k$ is a vector field (in subscript notation) and

$$dA_d = c_2(d, R) dA_2 \quad dV_D = c_3(D, R) dV_3. \tag{1.3}$$

Thus, we can rewrite the fractional integrals in (1.2) as conventional ones

$$\int_{\partial W} c_2(d, R) f_k n_k dA_2 = \int_W \nabla_k (c_2(d, R) f_k) dV_3, \tag{1.4}$$

and, effectively, deal with formulas in (conventional) Euclidean setting, provided we have the coefficients $c_3$ and $c_2$. Indeed, the latter are specified according to the fractional integral adopted. Tarasov defines them in the Riesz form and then it follows that

$$\begin{aligned} c_2(d, R) &= |\mathbf{R}|^{d-2} \frac{2^{2-d}}{\Gamma(d/2)}, \\ c_3(D, R) &= |\mathbf{R}|^{D-3} \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)}. \end{aligned} \tag{1.5}$$

Also, the following operators (or, generalized derivatives) are used

$$\begin{aligned} \nabla_k^D f &= c_3^{-1}(D, R) \frac{\partial}{\partial x_k} [c_2(d, R) f] \equiv c_3^{-1}(D, R) \nabla_k [c_2(d, R) f], \\ \left(\frac{d}{dt}\right)_D f &= \frac{\partial f}{\partial t} + c(D, d, R) v_k \frac{\partial f}{\partial x_k}, \end{aligned} \tag{1.6}$$

where $c(D,d,R) = c_3^{-1}(D,R)\,c_2(d,R)$.

Observe:

(i) Postulating $c_2$ and $c_3$ in (1.5), the fractional power law of mass is admitted, which forms the physical foundation of our approach. One can further note that, in practice, the volume $dV_3$ is not really infinitesimal but an upper length cutoff for the fractal structure. The lower one is given by whatever molecular scale in a specific problem. By appropriately modifying the forms of $c_2, c_3$, the asymptotic properties can be incorporated. Thus, the theory is suited for physical fractals − sometimes called pre-fractals − as opposed to mathematical fractals without any cutoffs. This is shown in Fig. 1.

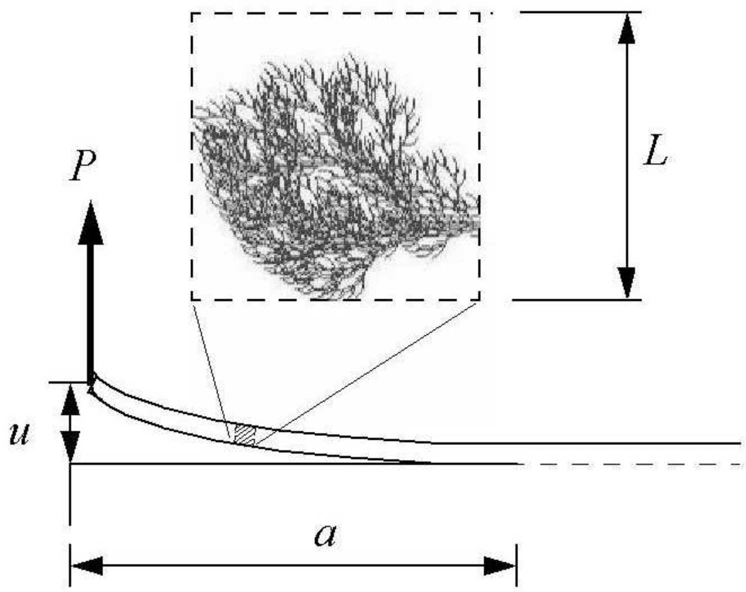


Figure 1. Fracture and peeling of a microbeam of thickness $L$ off a substrate. A representative volume element $dV_3$ imposed by the pre-fractal structure characterized by upper cutoff scale $L$ is shown. Thus, the beam is homogeneous above the length scale $L$. By introducing random variability in that structure, one obtains a random beam according to (4.6).

(ii) Mathematically, the theory involves fractional integrals but conventional derivatives plus the $c_3$ and $c_2$ coefficients, with the order of integrals being directly given by fractal dimensions of volumes ($D$) and surfaces ($d$). Having conventional derivatives makes the present theory easier to deal with than those with fractional derivatives.

(iii) In view of (i) the theory is limited to homogeneous fractal media. It may be extended to inhomogeneous (and, therefore, via extension to an ensemble) random media by following a standard procedure in stochastic solid mechanics [15].

(iv) The equations governing problems in one dimension cannot be consistently obtained from the equations governing problems in three dimensions. For example, the one dimensional fractal wave equation is not equivalent to that of a plane wave in three dimensions.

Point (iv) represents a drawback stemming from the fact that $c_2$ and $c_3$ are based on the Riesz measure which confines the analysis only to isotropic cases. That drawback can be removed by introducing a product measure instead, whereby a possible anisotropy is also incorporated, further ensuring that the mechanical approach to continuum mechanics is consistent with the energetic approach. To this end, note that, while the mass distribution in conventional continuum mechanics [16] is

$$d\mu(\mathbf{x}) = \rho(\mathbf{x})dV_3, \tag{1.7}$$

where $\rho(x)$ is mass density and $dV_3$ is the Lebesgue measure in $\mathbb{R}^3$, the product measure we now introduce is

$$d\mu_k(x_k) = \rho(\mathbf{x})c_1(\alpha_k, x_k)dx_k, \quad k = 1, 2, 3. \tag{1.8}$$

Thus, while (1.7) applies to a non-fractal mass distribution $M \sim x_1x_2x_3$, (1.8) applies to a fractal mass distribution $M \sim x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}$, the total fractal dimension being $D = \alpha_1 + \alpha_2 + \alpha_3$. For simplicity, here we adopt a form based on a Riemann-Liouville integral

$$c_1^{(k)} = \frac{|x_k|^{\alpha_k - 1}}{\Gamma(\alpha_k)}, \quad k = 1, 2, 3, \tag{1.9}$$

so as to replace (1.5) by

$$\begin{aligned} c_2^{(k)} &= c_1^{(i)}c_1^{(j)} = \frac{|x_i|^{\alpha_i - 1}|x_j|^{\alpha_j - 1}}{\Gamma(\alpha_i)\Gamma(\alpha_j)}, \quad i, j \neq k, \\ c_3 &= c_1^{(1)}c_1^{(2)}c_1^{(3)} = \frac{|x_1|^{\alpha_1 - 1}|x_2|^{\alpha_2 - 1}|x_3|^{\alpha_3 - 1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}. \end{aligned} \tag{1.10}$$

The operators (1.6) still apply, with $c = c_3^{-1}c_2$.

Continuing in the vein of the above developments, in this paper we extend fracture mechanics to materials described by $D$, $d$ and $R$. Since the fracture profile is usually observed to have an irregular and nonsmooth geometry, a methodology involving fractal dimensions and fractional integrals is expected to be physically more reasonable. In general, we study the energy aspects of fracture and, as an example, a beam made of a fractal material. In particular, we re-examine the crack driving force $G$ stemming from the Griffith fracture criterion extended to fractal media, considering either dead-load or fixed-grip conditions, derive the equations governing a fractally structured Timoshenko beam, then study peeling of a beam off a substrate. Finally, we extend the theory to random media, and examine the effects of ensemble averaging over random fractal beams.

## 2. Griffith's theory for elastic-brittle solids with fractal geometries

According to Griffith's theory of elastic-brittle solids [17], the strain energy release rate $G$ is given by

$$G = \frac{\partial W}{\partial A} - \frac{\partial U^e}{\partial A} = 2\gamma, \tag{2.1}$$

where $A$ is the crack surface area formed, $W$ is the work performed by the applied loads, $U^e$ is the elastic strain energy, and $\gamma$ is the energy required to form a unit of new material surface, e.g. [18]. The material parameter $\gamma$ is conventionally taken as constant, but, given the presence of a randomly microheterogeneous material structure, its random field nature is sometimes considered explicitly [1,2]. Recognizing that the random material structure also affects the elastic moduli (such as $E$), the computation of $U^e$ and $G$ in (2.1) also needs to be re-examined [19]. With reference to Fig. 2, we consider a 3-D material body described by $D$ and $d$, and having a crack of depth $a$ and a fractal dimension $DF$.

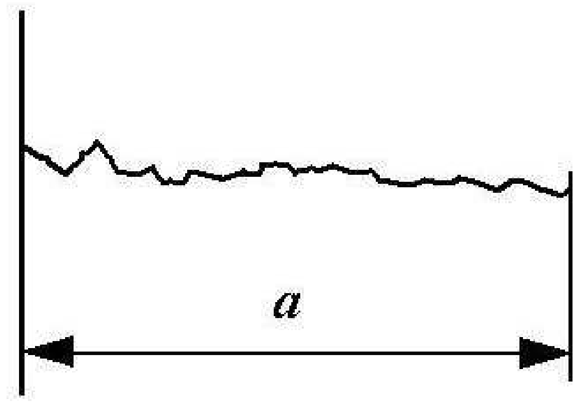


Figure 2. Showing a crack of depth $a$, with a fractal dimension $DF$.

The step taken here is to admit a fractal character of $\gamma$ and $E$; a generalization to a statistical ensemble, thereby admitting randomness, will be carried out in Section 4. Thus, focusing on a fractal porous material, we have

$$U^e = \int_W \rho\ u\ dV_D = \int_W \rho\ u\ c_3\ dV_3, \tag{2.2}$$

where $c_3$ is given by $(1.10)_2$. By revising Griffith's derivation for a fractal elastic material, we then obtain

$$U^e = \frac{\pi a^2 c_1^2 \sigma^2}{8\mu}(K+1)c_3, \tag{2.3}$$

with $\nu$ being the Poisson ratio, and

$$K = \begin{cases} 3-4\nu & \text{for plane strain} \\ \dfrac{3-\nu}{1+\nu} & \text{for plane stress} \end{cases} \tag{2.4}$$

the Kolosov constant.

**Dead-load conditions**. Equation (2.1) becomes

$$G = \frac{\partial U^e}{\partial A} = 2\gamma. \tag{2.5}$$

If $A = 2a \times 1$, this gives the critical stress

$$\sigma_c = \sqrt{\frac{2\gamma E}{(1-\nu^2)\pi a \ c_3 \ c_1^2}}. \tag{2.6}$$

However, if the fracture surface is fractal of a fractal dimension $DF$, then we should use $\partial/\partial l_{DF}$ instead of $\partial/\partial a$. Now, since we have (note Fig. 2)

$$dl_{DF} = c_1(DF, a)da, \tag{2.7}$$

the new partial derivative becomes

$$\frac{\partial}{\partial l_{DF}} = \frac{\partial}{c_1 \partial a}, \tag{2.8}$$

where

$$c_1(x, DF) = \frac{|x|^{DF-1}}{\Gamma(DF)}. \tag{2.9}$$

As a result,

$$\sigma_c = \sqrt{\frac{2\gamma E}{(1-\nu^2)\pi a \ c_3 \ c_1}}. \tag{2.10}$$

**Fixed-grip conditions**. We consider the case of a crack of depth $a$ and width $B$ in plane strain. In this case the displacement is constant (i.e., non-random), and the load is random. Now, only the first term in (2.1) remains, so that

$$G = -\frac{\partial U^e (a)}{B \partial l_{DF}} = -\frac{\partial U^e (a)}{B c_1 \partial a}. \tag{2.11}$$

## 3. Timoshenko beam with a fractal geometry

First we recall that a Timoshenko beam has two degrees of freedom $(q_1, q_2)$ at each point: the transverse displacement $q_1 = w$ and the rotation $q_2 = \varphi$. Given its fractal structure (of dimension $D$) in the $x$ direction, which can readily be measured by image analysis, the $dx$ element is replaced by

$$dl_D = c_1(D, x)dx, \tag{3.1}$$

where, by the argument leading to (2.9), $c_1(x, D) = |x|^{D-1}/\Gamma(D)$.

The kinetic energy, which in the non-fractal case is (with $\dot{\varphi} \equiv \partial\varphi/\partial t$)

$$T = \frac{1}{2}\rho_0 \int_0^l \left[I(\dot{\varphi})^2 + A(\dot{w})^2\right] dx, \tag{3.2}$$

in view of (3.1), gets modified to

$$T = \frac{1}{2}\rho_0 \int_0^l \left[I(\dot{\varphi})^2 + A(\dot{w})^2\right] dl_D. \tag{3.3}$$

The potential energy, which in the non-fractal case is (with $\varphi,_x \equiv \partial\varphi/\partial x$)

$$U = \frac{1}{2}\int_0^l \left[EI(\varphi,_x)^2 + \kappa\mu A(w,_x - \varphi)^2\right] dx, \tag{3.4}$$

again in view of (3.1), gets modified to

$$\begin{aligned} U &= \frac{1}{2}\int_0^l \left[EI\left(\frac{\partial\varphi}{\partial l_D}\right)^2 + \kappa\mu A\left(\frac{\partial w}{\partial l_D} - \varphi\right)^2\right] dl_D \\ &= \frac{1}{2}\int_0^l \left[EIc_1^{-2}(\varphi,_x)^2 + \kappa\mu A\left(c_1^{-1}w,_x - \varphi\right)^2\right] c_1 dx. \end{aligned} \tag{3.5}$$

Assuming null external distributed load and moment, the Lagrangian of the beam system is

$$\begin{aligned} L = T - U = \frac{1}{2}\int_0^l &\left\{\rho_0\left[I(\dot{\varphi})^2 + A(\dot{w})^2\right] c_1\right. \\ &\left. - \left[EIc_1^{-2}(\varphi,_x)^2 + \kappa\mu A\left(c_1^{-1}w,_x - \varphi\right)^2\right]\right\} c_1 dx. \end{aligned} \tag{3.6}$$

Next, the Euler–Lagrange equations

$$\frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{q}_i}\right] + \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left[\frac{\partial L}{\partial(q_{i,j})}\right] - \frac{\partial L}{\partial q_i} = 0 \tag{3.7}$$

imply these equations of elastodynamics governing the fractal Timoshenko beam

$$\begin{aligned} \rho_0 A c_1 \ddot{w} &= \left[\kappa\mu A\left(c_1^{-1}w,_x - \varphi\right)\right],_x, \\ \rho_0 I c_1 \ddot{\varphi} &= \left(EIc_1^{-1}\varphi,_x\right),_x + \kappa\mu A c_1\left(c_1^{-1}w,_x - \varphi\right). \end{aligned} \tag{3.8}$$

Following the definition of generalized derivatives (1.6) (in 1D case: $\nabla_x^D f = c_1^{-1} f,_x$), the above equation has the form

$$\begin{aligned} \rho_0 A \ddot{w} &= \nabla_x^D\left[\kappa\mu A\left(\nabla_x^D w - \varphi\right)\right], \\ \rho_0 I \ddot{\varphi} &= \nabla_x^D\left(EI\nabla_x^D\varphi\right) + \kappa\mu A\left(\nabla_x^D w - \varphi\right). \end{aligned} \tag{3.9}$$

Note that the beam equation can also be derived from force and moment balance analysis, which should be consistent with (3.9). To start the development, we recall the expressions of shear force ($V$) and bending moment ($M$) in the non-fractal case

$$V = \kappa\mu A\left(w_{,x} - \varphi\right), \qquad M = -EI\varphi_{,x}\,. \tag{3.10}$$

In view of (3.1), this gets modified to

$$V = \kappa\mu A\left(\nabla_x^D w - \varphi\right), \qquad M = -EI\nabla_x^D\varphi. \tag{3.11}$$

The balance equations of force and moment in the non-fractal case give

$$\rho_0 A\ddot{w} = V_{,x}\,, \qquad \rho_0 I\ddot{\varphi} = V - M_{,x}\,, \tag{3.12}$$

which, again on account of (3.1), this gets modified to

$$\rho_0 A\ddot{w} = \nabla_x^D V, \qquad \rho_0 I\ddot{\varphi} = V - \nabla_x^D M. \tag{3.13}$$

Substituting (3.11) into (3.13), we obtain (3.9) again. In other words, the mechanical (Newtonian) approach is consistent with the energetic (Lagrangian functional) approach. On the other hand, we note that the replacement of the derivative $f_{,x}$ by $\nabla_x^D f = c_1^{-1} f_{,x}$ is necessary for introducing the fractional integral of mass, or the results under such two approaches will not be equivalent. The underlying physical mechanism may lie in the fact that the fractional power law of mass implies a fractal dimension of scale measure, so the derivatives involving spatial scales should be modified to incorporate such effect by postulating $c_1 \sim c_3$.

In the case of elastostatics and when the rotational degree of freedom ceases to be independent ($\varphi = \partial w/\partial l_D = \nabla_x^D w$), we find the equation of a fractal Euler–Bernoulli beam

$$\nabla_x^D \nabla_x^D \left(EI\nabla_x^D \nabla_x^D w\right) = 0, \tag{3.14}$$

This shows that

$$M = EI\nabla_x^D \nabla_x^D w. \tag{3.15}$$

The relationship between the bending moment ($M$) and the curvature ($\nabla_x^D \nabla_x^D w$) still holds, while $c_1$ enters the determination of curvature ($\nabla_x^D \nabla_x^D w = c_1^{-1}\left(c_1^{-1} w_{,x}\right)_{,x}$).

## 4. Peeling a layer off a substrate

**Dead-load conditions**. Here we return to the system in Fig. 1. The case of constant load implies that the force is prescribed, and only the second term in (2.1) remains. Assuming an Euler–Bernoulli beam, the strain energy is

$$U(a) = \int_0^a \frac{M^2}{2IE} dx, \tag{4.1}$$

where $a$ is crack length, $M$ is bending moment, $I$ is beam's moment of inertia, and $E$ is its elastic modulus. Henceforth, we simply work with $a = A/B$, where $B$ is the constant beam (and crack) width. In view of Clapeyron's theorem, the strain energy release rate may be written as

$$G = \frac{\partial U}{B\partial a}. \tag{4.2}$$

For a layer modeled as a fractal Euler–Bernoulli beam whose equation was derived in the previous section, we have

$$U(a) = \int_0^a \frac{M^2}{2IE} dl_D = \int_0^a \frac{M^2}{2IE} c_1 dx, \tag{4.3}$$

so that

$$G = \frac{\partial U}{c_1 B\partial a}. \tag{4.4}$$

Now, if the beam's material is random, $E$ is a random field parametrized by $x$, which we can write as a sum of a constant mean $\langle E\rangle$ and a zero-mean fluctuation $E'(x)$

$$E(\omega, x) = \langle E\rangle + E'(\omega, x) \qquad \omega \in \Omega, \tag{4.5}$$

where $\Omega$ is a sample space. Take $E'(x,\omega)$ as a wide-sense stationary random field. A random material is thus defined as an ensemble

$$\mathcal{B} = \{B(\omega); \omega \in \Omega\} = \{E(\omega, x); \omega \in \Omega, x \in [0, a]\}. \tag{4.6}$$

Here, and in the following, we explicitly show the dependence on $\omega$, whenever we wish to indicate the random nature of a given quantity prior to ensemble averaging.

Clearly, $U$ is a random integral, such that, for each and every realization $\omega \in \Omega$, we should consider

$$U(a, E(\omega)) = \int_0^a \frac{M^2 c_1 dx}{2IE(\omega, x)}. \tag{4.7}$$

Upon ensemble averaging, this leads to an average energy

$$\langle U(a, E)\rangle = \left\langle \int_0^a \frac{M^2 c_1 dx}{2I\left[\langle E\rangle + E'(\omega, x)\right]} \right\rangle. \tag{4.8}$$

In the conventional formulation of deterministic fracture mechanics, random microscale heterogeneities $E'(x,\omega)$ are disregarded, and (4.7) is evaluated by simply replacing the denominator by $\langle E\rangle$, so that

$$U(a, \langle E\rangle) = \int_0^a \frac{M^2 c_1 dx}{2I\langle E\rangle}. \tag{4.9}$$

Clearly, this amounts to postulating that the response of an idealized homogeneous material is equal to that of a random one on average. To make a statement about $\langle U(a, E)\rangle$ versus $U(a, \langle E\rangle)$, and about $\langle G(E)\rangle$ versus $G(\langle E\rangle)$, first, note the random field $E$ is positive-valued almost surely. Then, Jensen's inequality yields an inequality between harmonic and arithmetic averages of the random variable $E(\omega)$

$$\frac{1}{\langle E\rangle} \le \left\langle \frac{1}{E} \right\rangle. \tag{4.10}$$

whereby the $x$-dependence is immaterial in view of the assumed wide-sense stationary of field $E$. With (4.8) and (4.9), and given that the conditions required by Fubini's theorem are met, this implies that

$$\begin{aligned} U(a, \langle E \rangle) &= \int_0^a \frac{M^2 c_1 dx}{2I \langle E \rangle} \leq \int_0^a \frac{M^2 c_1}{2I} \left\langle \frac{1}{E} \right\rangle dx \\ &= \left\langle \int_0^a \frac{M^2 c_1 dx}{2IE(\omega, x)} \right\rangle = \langle U(a, E) \rangle \,, \end{aligned} \tag{4.11}$$

Now, defining the strain energy release rate $G(a, \langle E \rangle)$ in a hypothetical material specified by $\langle E \rangle$, and the strain energy release rate $\langle G(a, E) \rangle$ properly ensemble averaged in the random material $\{E(\omega, x); \omega \in \Omega, x \in [0, a]\}$

$$G(a, \langle E \rangle) = \frac{\partial U(a, \langle E \rangle)}{B c_1 \partial a} \qquad \langle G(a, E) \rangle = \frac{\partial \langle U(a, E) \rangle}{B c_1 \partial a}, \tag{4.12}$$

and noting that the side condition is the same in both cases

$$U(a, \langle E \rangle) \mid_{a=0} = 0 \qquad \langle U(a, E) \rangle \mid_{a=0} = 0, \tag{4.13}$$

we find

$$G(a, \langle E \rangle) \leq \langle G(a, E) \rangle \,. \tag{4.14}$$

This provides a formula for the ensemble average $G$ under dead-load conditions using deterministic fracture mechanics for Euler–Bernoulli beams made of fractal random materials.

Just like in the case of non-fractal materials [19], the inequality (4.14) shows that $G$ computed under the assumption that the random material is directly replaced by a homogeneous material ($E(x, \omega) = \langle E \rangle$), is lower than $G$ computed with $E$ taken explicitly as a spatially varying material property.

**Fixed-grip conditions**. On account of (2.11), assuming that there is loading by a force $P$ at the tip, we obtain

$$G = -\frac{u}{2Bc_1} \frac{\partial P}{\partial a}. \tag{4.15}$$

Take now a cantilever beam problem implying $P = 3uEI/(c_1 a)^3$. Then, we find

$$\langle G \rangle = -\frac{u}{2Bc_1} \left\langle \frac{\partial P}{\partial a} \right\rangle = -\frac{u}{2Bc_1} \frac{\partial \langle P \rangle}{\partial a} = \frac{9u^2 I \langle E \rangle}{2B(c_1 a)^4}. \tag{4.16}$$

Since the load $-$ be it a force and/or a moment $-$ is always proportional to $E$, this indicates that $G$ can be computed by direct ensemble averaging of $E$ under fixed-grip loading, and, indeed, the same conclusion carries over to Timoshenko beams.

The foregoing analysis may be extended to (i) fractal Timoshenko beams, (ii) mixed-loading conditions and (iii) stochastic crack stability by generalizing the study of non-fractal, random beams carried out in [19].

## 5. Closure

While several different approaches to fracture of fractal materials were developed since the nineties, the present method has its particular advantages. Specifically, (i) it is suited to deal with pre-fractals as opposed to mathematical fractals without any cutoffs; (ii) it involves fractional integrals of the order directly related to fractal dimensions of volumes and surfaces but conventional derivatives; (iii) it can handle inhomogeneous and/or anisotropic fractal media; (iv) the equations governing the problems in one dimension can be consistently obtained from the equations governing the problems in three dimensions.

Our study is confined to fractal, elastic-brittle materials, with focus being on the determination of continuum mechanical equations accounting for the fractal dimension of mass distribution ($D$), the fractal dimension of surface ($d$) and the resolution lengthscale $R$. We study the energy aspects of fracture and re-examine the crack driving force $G$ stemming from the Griffith fracture criterion extended to fractal media. We also derive equations governing a Timoshenko beam with a fractal geometry, and specialize it to a Bernoulli-Euler beam. On that basis, we then consider either dead-load or fixed-grip conditions in peeling of a beam off a substrate. Finally, we generalize the model to random fractal beams, and study the effects of ensemble averaging.

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